PHYSICS DEPARTMENT
PRINCETON UNIVERSITY

GRADUATE PRELIMINARY EXAMINATION

Monday, January 8, 2001 - 9:00 am - 12:00 noon

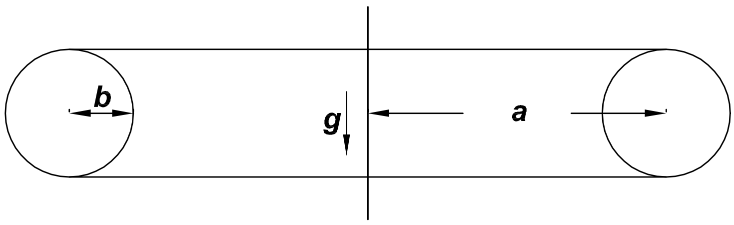
Part I.

Answer two out of the three questions in Section A (Mechanics) and two out of the three questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Monday, January 8, 2001 - 9:00 am - 12:00 noon

Part I. Section A. Mechanics



1. Find the frequency of small oscillations about uniform circular motion of a point mass that is constrained to move on the surface of a torus (donut) of major radius a and minor radius b whose axis is vertical.

Monday, January 8, 2001 - 9:00 am - 12:00 noon

Part I. Section A. Mechanics (continued)

The following two problems relate to a calculation of the angular frequency Ω of free precession of a planet or star whose angular frequency of rotation about its axis is ω. The problems themselves are independent.

2. Suppose that the density ρ of the object is uniform, and that its shape can be determined by the condition of hydrostatic equilibrium. Deduce an expression for the (small) quantity $\epsilon(\omega, M, r_P)$ that relates the equatorial radius r_E to the polar radius r_P by the form $r_E = r_P(1+\epsilon)$, where $M \approx 4\pi\rho r_P^3/3$ is the mass of the object.

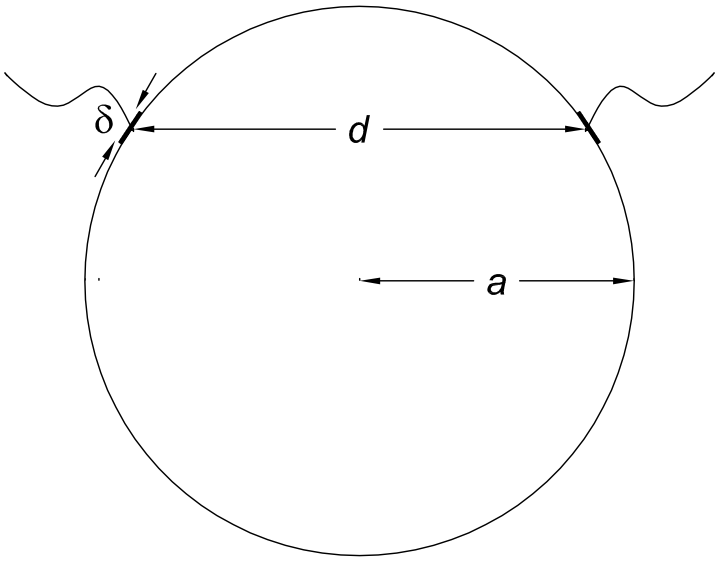
3. Suppose the object can be treated as a rigid body whose principal moments of inertia obey $(I_P - I_E)/I_P = \epsilon$ to deduce the angular frequency Ω of free precession in terms of the angular frequency ω of rotation.

The reader will note that the models of the two problems are somewhat contradictory. However, they work fairly well for the Earth, whose observed free precession period of 430 days (Chandler, 1891) is about 1.6 times that as estimated above. The Chandler wobble is thought to be driven by surface wind and water; see Science **289**, 710 (4 Aug. 2000). First evidence for free precession of a pulsar, PSR B1828-11, has recently been reported by Princeton Ph.D. I.H. Stairs, Nature **406**, 484 (2000), with a period about 1/150 that of the above model. This discrepancy is ascribed to little understood aspects of the superfluid interior of the pulsar.

Monday, January 8, 2001 - 9:00 am - 12:00 noon

Part I. Section B. Electricity and Magnetism

1. Calculate the resistance between two contacts on the rim of a disk of radius a, thickness $t \ll a$, and conductivity σ, when each (perfectly conducting) contact extends for a small distance δ around the circumference, and the distance along the chord between the contacts is $d \gg \delta$.



Hint: The contacts set up semicircular regions of radius $\delta/2$ of nearly uniform potential that extend into the resistive disk.

Monday, January 8, 2001 - 9:00 am - 12:00 noon

Part I. Section B. Electricity and Magnetism (continued)

2. A betatron is a device in which ultrarelativistic electrons are held in a circle of fixed radius R (taken to be centered on the origin in the x-y plane) by a magnetic field $B_z(r,t)$ while their energy is increased via a changing magnetic flux $d\Phi/dt = \pi R^2 dB_{z,\mathrm{ave}}/dt$ through the circle. Motion of the electrons perpendicular to the circle is prevented by means that need not be considered here.

Deduce the relation between the magnetic field B_z at radius R and the magnetic field $B_{z,\mathrm{ave}}$ averaged over the area of the circle. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of B_z, $dB_{z,\mathrm{ave}}/dt$ and R.

Hints: The electrons in this problem are ultrarelativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2 \gg 1$, where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.

Monday, January 8, 2001 - 9:00 am - 12:00 noon

Part I. Section B. Electricity and Magnetism (continued)

3. A plane electromagnetic pulse $E(z,t) = f(z/c - t)$ is incident from vacuum at $z < 0$ on a dielectric medium that extends from $z = 0$ to $z = a$. The region $z > a$ is also vacuum. The pulse has large Fourier components only at frequencies near its central angular frequency ω_0. The index of refraction $n(\omega)$ of the medium is near unity, so reflections at the boundaries can be ignored, and the approximation

$$\omega n(\omega) \approx \omega_0 + \left.\frac{d(\omega n(\omega))}{d\omega}\right|_{\omega_0} (\omega - \omega_0)$$

holds over the relevant frequency bandwidth of the waveform,

Compute the waveform $E(z,t)$ in the dielectric region $0 < z < a$, and in the vacuum region $a < z$.

If ω_0 is chosen to lie between two spectral lines of the medium, which is pumped by lasers at those frequencies into inverted populations, then the group velocity $v_g(\omega_0)$ can be negative, as recently demonstrated by Wang *et al.*, Nature **406**, 277 (2000). Comment on any unusual features of the pulse propagation in this case.

Hint: first discuss the propagation of a monochromatic wave; then consider its implications for the Fourier analysis of the pulse.