Department of Physics, Princeton University

Graduate General Examination
Part III

Monday, January 7, 2002
9:00 am - 12:00 noon

This part of the General Examination poses THREE questions on Condensed Matter Physics and THREE on Elementary Particles and Nuclear Physics. Answer THREE questions, at least ONE from each section.

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Condensed Matter Physics

1. A linear chain is formed from N atoms with spacing a. The position of the n-th atom is given by $\mathbf{R}_n = na\hat{\mathbf{x}}$. The unperturbed wave function and energy at the site $\mathbf{R}_n$ are $|n\rangle$ and ϵ^0, respectively, i.e., the atomic wave function is $\phi(\mathbf{r} - \mathbf{R}_n) \equiv \langle \mathbf{r}|n\rangle$.

   We assume that the electrons are *spinless* and non-interacting. In the tight-binding approximation, the Hamiltonian of the chain is

$$H = \sum_n \epsilon^0 |n\rangle\langle n| \; - \; t_0 \; (\; |n\rangle\langle n+1| \; + \; |n\rangle\langle n-1| \;) \, . \tag{1}$$

   The first term is the on-site energy. The second and third terms represent hops to the left and right neighbors, respectively, with t_0 (the "hopping" integral) defined by $t_0 = -\langle n|v|n+1\rangle$, where v is the atomic potential.

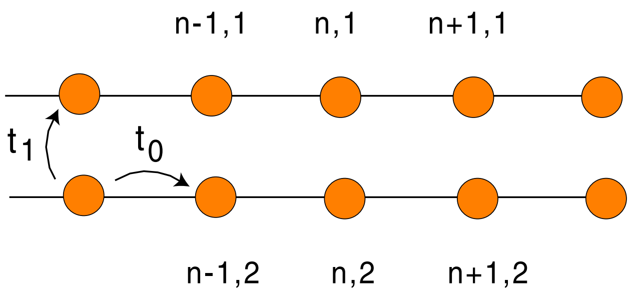
   The eigenstate $\psi_{\mathbf{k}}(\mathbf{r}) = \langle \mathbf{r}|\mathbf{k}\rangle$ of H may be expanded in terms of $|n\rangle$ as

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N}} \sum_n c_n |n\rangle$$

   (we assume that $\langle n|n'\rangle = \delta_{n,n'}$).

   (a) Use Bloch's theorem (or other reasoning) to determine the coefficients c_n.

   (b) Verify that H is diagonalized in the basis $\{|\mathbf{k}\rangle\}$, i.e., $\langle \mathbf{k}|\hat{H}|\mathbf{k}'\rangle = \epsilon(\mathbf{k})\delta_{\mathbf{k},\mathbf{k}'}$, and find the energy $\epsilon(\mathbf{k})$ of the Bloch state $|\mathbf{k}\rangle$.



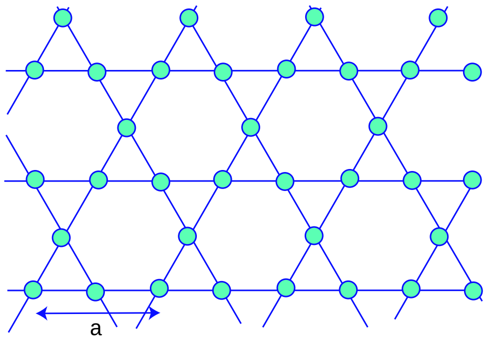
   Two parallel chains $\alpha = 1, 2$. The atoms are indexed by n, α. The hopping integrals t_0 and t_1 between nearest neighbors are indicated.

   (c) Suppose we have two identical chains side-by-side, as in the figure above. The atomic states are now $|n, \alpha\rangle$ where $\alpha = 1, 2$ is the chain index. Let the hopping integral between the chains be $t_1 = -\langle n, 1|v|n, 2\rangle$. By generalizing eq. (1) slightly, obtain the Hamiltonian H_2 of the 2-chain system.

   Be sure that your H_2 is Hermitian.

   (d) Diagonalize H_2 to obtain the energy $\epsilon_\alpha(k)$ of the two bands.

2. The figure below shows the two-dimensional (2D) Kagome lattice. All the atoms are identical and the angle between any pair of bonds is either $\pi/3$ or $2\pi/3$.



(a) How many atoms are in each repeating unit (the 'basis')? Find the Bravais lattice and its spanning vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

(b) What are the reciprocal lattice vectors $\mathbf{b}_i$?

(c) The Bragg spots fall into 2 sets (bright and dim) with intensities I_1 and I_2, respectively. What are the ratios I_1/I_0 and I_2/I_0, where I_0 is the forward-scattering intensity ($\mathbf{q} = \mathbf{0}$)? Make a reasonably accurate sketch of the reciprocal lattice, identifying the bright and dim sets of Bragg spots that would be observed in an x-ray scattering experiment.

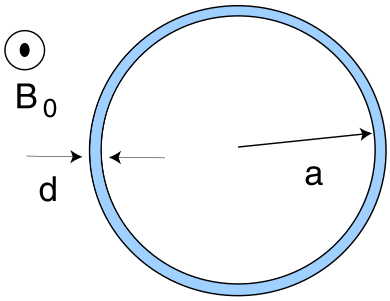
(d) If an additional atom (per basis) is inserted at an empty site, the entire set of dim Bragg spots can be extinguished. Where is the site? What is the new reciprocal lattice, and the new structure factor S'?

3. The Ginzburg-Landau (GL) expansion for the free energy of a superconductor is

$$\Delta F = \int d^3r \left[a(T)|\psi|^2 + \frac{1}{2}b|\psi|^4 + \frac{1}{2m}\left|\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\psi\right|^2 + \frac{B^2}{2\mu_0}\right], \tag{1}$$

where $a(T) \equiv \alpha(t-1)$ is linear in the reduced temperature $t = T/T_{c0}$ with T_{c0} the critical temperature in zero magnetic field. The parameters α and b are positive constants, and $e^* = 2e$ ($m, e, \hbar$ and μ_0 are the familiar universal constants). $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field in the sample. The amplitude squared of the GL wave function $\psi = |\psi|\mathrm{e}^{i\theta(\mathbf{r})}$ equals the superfluid density, *i.e.*, $n_s = |\psi|^2$, while its phase determines the supercurrent density $\mathbf{J}_s$ (or superfluid velocity $\mathbf{v}_s$):

$$\mathbf{J}_s = n_s e^* \mathbf{v}_s, \qquad \mathbf{v}_s = \frac{1}{m}[\hbar\nabla\theta - e^*\mathbf{A}]. \tag{2}$$



End-view of a long, thin-walled superconducting cylinder in a magnetic field $\mathbf{B}_0$.

Consider a long superconducting cylinder (shown end-on in the figure above) of radius a and wall thickness $d \ll a$. An external magnetic field $\mathbf{B}_0$ is applied parallel to the axis as shown. Assume that $d \ll \xi$ and $d \ll \lambda$, where ξ and λ are the GL coherence length and magnetic penetration length, respectively. In this limit, you may take $|\psi|$ and the magnetic field inside the superconductor to be *uniform*.

(a) Simplify eq. (1) in this limit, and express ΔF in terms of $\mathbf{v}_s$.

(b) Assume $B_0 = 0$. Find the equilibrium value of ψ at a temperature $T < T_{c0}$. What is $\mathbf{v}_s$?

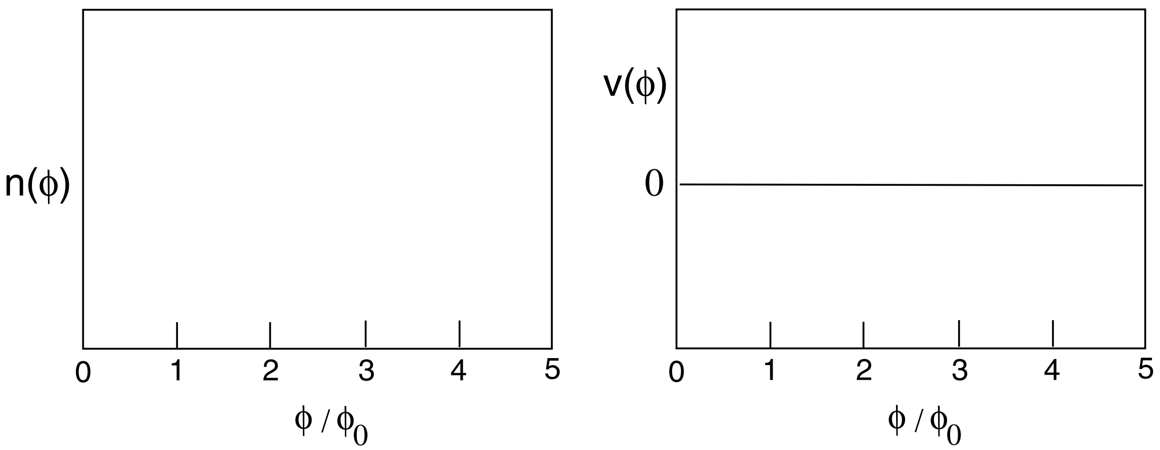
(c) Next, we turn on the field. If the flux enclosed by the cylinder is ϕ, show that the magnitude of $\mathbf{v}_s$ is given by

$$v_s = \frac{\hbar}{ma}\left[n - \frac{\phi}{\phi_0}\right],$$

where n is an integer. Find ϕ_0.

(d) Explain briefly why ϕ is not quantized (as in the case of a thick cylinder).

(e) Calculate the equilibrium value of $|\psi|^2$ and the corresponding value of n as a function of ϕ. Plot the variation of $n(\phi)$ and $v_s(\phi)$ in the panels below.

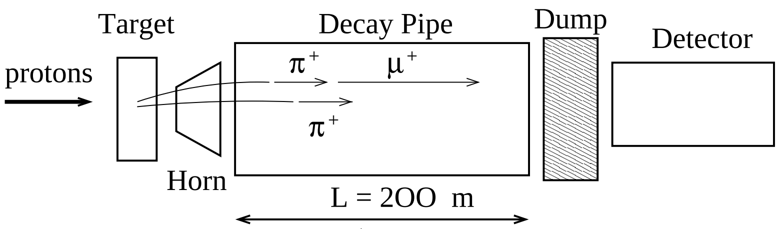


[*Hint*: ΔF is a minimum at equilibrium.]

For recent developments, see Y. Liu *et al.*, Science **294**, 2332 (14 Dec. 2001).

Section B. Elementary Particles and Nuclear Physics

1. A typical high-energy neutrino beam is made from the decay of π mesons that have been produced in proton interactions on a target, as sketched in the figure below.



Assume the magnetic "horn" collects a pure π^+ beam into the decay pipe. The main source of neutrinos comes from the decay $\pi^+ \to \mu^+ \nu_\mu$. Facts: $m_\pi = 139.6$ MeV/c^2, $m_\mu = 105.7$ MeV/c^2. In this problem, neutrinos can be taken as massless. The spin and parity quantum numbers for a π meson are $J^P = 0^-$.

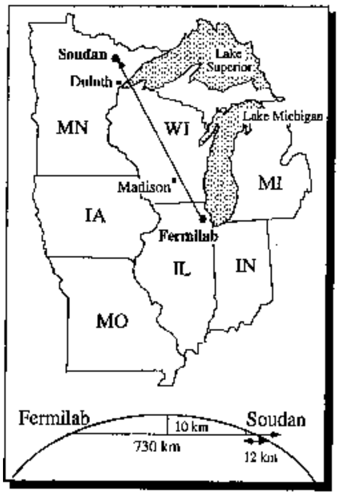
(a) What other types of neutrinos than ν_μ are produced in the decay pipe?

(b) If the decay pions have energy $E_\pi \gg m_\pi$, what is the characteristic angle θ_C of the decay neutrinos with respect to the direction of the π^+?

(c) Compute the energy of the neutrino E_ν^* in the pion rest frame and derive the relationship between $\cos\theta$, the neutrino decay angle relative to the pion direction of flight, and $\cos\theta^\star$, the decay angle in the rest frame of the pion. Note that $\cos\theta^*$ can be written in terms of $\lambda \equiv E_\nu / E_\nu^*$ and $\cos\theta$.

(d) Show that when observing the neutrino beam energy spectrum at a fixed lab angle θ with respect to the π meson direction, there is an upper limit on the value of E_ν independent of E_π.

(e) Start with the angular distribution of the neutrinos in the pion rest frame and boost this into the lab frame, using the axis defined by the π meson flight direction ($\theta = 0$). At what lab angle θ_K, in terms of λ, is there a large off-axis enhancement in the observed rate of neutrinos?



(f) What is the minimum π^+ energy E_π needed to produce an off-axis enhanced neutrino beam of $E_\nu = 1.5$ GeV? The MINOS detector is located in the Soudan mine 730 km from the decay pipe at an angle $\theta = 0$ with respect to the π^+ meson beam. How far away from Soudan normal to the beam direction should you build a detector to observe the off-axis peak of $E_\nu = 1.5$ GeV neutrinos?

2. The Lagrangian for the charged current (CC) interaction of quarks with the W bosons has the form

$$L^{\rm CC} = \frac{g}{\sqrt{2}} \left(\left(\, \bar{u} \; \bar{c} \; \bar{t} \, \right)_L \gamma^\mu W^+_\mu V_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L + \left(\, \bar{d} \; \bar{s} \; \bar{b} \, \right)_L \gamma^\mu W^-_\mu V^\dagger_{\rm CKM} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L \right) ,$$

where the Cabibbo-Kobayashi-Maskawa matrix, $V_{\rm CKM}$, is a 3×3 unitary matrix of coupling constants,

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} ,$$

that you may approximate as being real numbers with the exception of $V_{td} = \tilde{V}_{td} + i\Delta V_{td}$ and $V_{ub} = \tilde{V}_{ub} + i\Delta V_{ub}$, which have imaginary terms.

Facts: The quark content of the mesons are $B^0_d = (\bar{b}d)$, $\pi^+ = (\bar{d}u)$, and $\pi^- = (\bar{u}d)$.

(a) The B^0_d meson is observed to oscillate into a $\bar{B}^0_d$ meson. Sketch the quark-level Feynman diagrams for this process and indicate the V_{CKM} coupling constants at the vertices. Assume for simplicity that only t-quarks participate in internal quark lines. From the diagrams, or otherwise, determine the dependence of the matrix element M for the mixing process $B^0_d \to \bar{B}^0_d$ on the V_{CKM} coupling constants.

(b) Sketch the Feynman diagrams for $\bar{B}^0_d \to B^0_d$ and indicate the V_{CKM} coupling constants at the vertices. The matrix element derived from these diagrams is labeled $M^\dagger$ as it describes the conjugate process in the original Lagrangian, namely, the inverse of the process from part a). Consider also the matrix element M_{CP} for the process $\bar{B}^0_d \to B^0_d$ obtained by applying the combined operations of charge conjugation (C) and space inversion (parity, P) to the matrix element M for $B^0_d \to \bar{B}^0_d$. Explain why $M_{CP} \neq M^\dagger$ and show the dependence of the ratio $M_{CP}/M^\dagger$ on the V_{CKM} coupling constants

(c) Perform a similar analysis for the tree-level decay transition for the process $B^0_d \to \pi^+\pi^-$, comparing the CP transformed matrix element for this process with the matrix element for the process $\bar{B}^0_d \to \pi^+\pi^-$, ignoring mixing.

(d) Phases in processes describable with a single diagram cannot be measured directly. Start with an initially pure B^0_d meson state and explain qualitatively (without calculations) how the combination of mixing and decay can result in a physical observable of CP violation.

(e) Use one of the unitarity constraints on V_{CKM} and your knowledge of the charged current interaction to numerically approximate the ratio $\Delta V_{ub}/\Delta V_{td}$.

3. An experiment to search for proton decay can be carried out using a large tank of water as the proton source. The possible decay mode $p \to e^+ \pi^0$ is to be detected by Čerenkov light emitted when electromagnetic showers from the decay products traverse the water. Water has index of refraction 1.33 and radiation length 36 cm.

   (a) Estimate the total charged track length integral of the showers in the event and hence the total number of photons emitted in the visible region. Note that in water the energy loss of a relativistic charged particle due to Čerenkov radiation is approximately 400 eV/cm.

   (b) Deduce how big the water tank should be in order to contain the showers.

   (c) The light is to be detected by an array of photomultiplier tubes placed on the tank walls. If the optical transmission of the water is 20% and the photocathode efficiency is 15%, what fraction of the wall surface must be covered by photocathode to give a root-mean-square energy resolution of 10%?

   (d) The Super-Kamiokande detector searched for proton decay via $p \to e^+ \pi^0$ over a period of 414 days. No candidate decays were observed. From this data, a lower limit on the partial lifetime of the proton $\tau / Br\{p \to e^+ \pi^0\}$ was set to be 1.6×10^{33} years at a 90% confidence level. Make a rough estimate of the mass of the water in kilograms in the Super-K detector based on the lower limit set for the proton lifetime. Assume the expected background rate was negligible.