

PHYSICS DEPARTMENT  
PRINCETON UNIVERSITY

GRADUATE GENERAL EXAMINATION

Tuesday, January 8, 2002 - 9:00 am - 12:00 noon

Part IV.

This part of the General Examination poses SIX questions, TWO on Relativity and FOUR on General and Atomic Physics. You must do ONE relativity problem and TWO General and Atomic questions.

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

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Part IV. Section A. Relativity

1. Consider flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(dr^2 + r^2 d\Omega^2),$$

where  $a = 1/(1+z)$  is the expansion factor,  $z$  redshift,  $t$  proper time and  $r$  comoving radial coordinate. Friedman equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho$$

where  $\rho$  is total density of all the components and the derivative is with respect to proper time  $t$ . The continuity equation for each component is

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0,$$

where  $p_i$  is the pressure of  $i$ -th component and  $\rho_i$  its density. Assume we have matter component with  $p_m = 0$ ,  $\rho_m \propto a^{-3}$  and a dark energy or quintessence component with equation of state  $p_Q/\rho_Q = w$ .

- a) Find how quintessence energy density  $\rho_Q$  scales with redshift for arbitrary  $w$ . For which values of  $w$  will it dominate in the future?
- b) Solve for  $a(t)$  in the future assuming quintessence dominates. Show that for  $w < -1$  expansion factor  $a$  diverges at a finite time.
- c) Show that for  $w < -1/3$  there is a future horizon by computing the comoving distance light can propagate to in infinite future.

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Part IV. Section A. Relativity (continued)

Problem 1. (continued)

- d) Imagine there is a characteristic comoving scale  $\Delta r$  imprinted onto the matter fluctuations in the universe (such a scale could be caused for example by baryonic effects or by matter-radiation transition). Imagine such a scale is observed in redshift space where its size is  $\Delta z$  (this could be achieved for example by observing quasar absorption spectra). Find  $\Delta z/\Delta r$  as a function of redshift  $z$  for a universe containing matter and quintessence (assume  $\Delta z/z$  is small), expressing it in terms of matter and quintessence densities today. Evaluate the ratio of  $\Delta z$  for a given  $\Delta r$  between  $w = -2/3$  and  $w = -1$  models at  $z = 2$ . Assume the ratio of matter and quintessence densities today is  $\rho_{m0}/\rho_{Q0} = 1/2$ .

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Part IV. Section A. Relativity (continued)

2. Recent developments in string theory involve field theories with conformal symmetry. In this problem you are asked to derive some basic properties of conformal transformations and of conformal field theories and to show that classical electromagnetism is conformally invariant.

Consider a four-dimensional spacetime with a flat metric  $g_{ij} = \eta_{ij}$ . Poincaré transformations are the most general coordinate transformations that preserve distances, *i.e.*, the metric  $g_{ij}$  is invariant under them. A more general set of transformations are conformal transformations. These transformations preserve angles but not lengths, *i.e.* the metric transforms as  $g'_{ij}(x') = \Omega(x)g_{ij}(x)$ . Consider the infinitesimal coordinate transformation,

$$x^{i'} = x^i - f^i(x), \quad (1)$$

where  $f^i(x)$  is considered infinitesimal.

- a) Show that the transformation (1) is a conformal transformation provided  $f^i$  satisfies the equation

$$\partial_i f_j + \partial_j f_i - \frac{1}{2}(\partial^k f_k)\eta_{ij} = 0, \quad (2)$$

where indices are lowered with  $\eta_{ij}$ . Show that

$$f_i = a_i + \omega_{ij}x^j + ax^i + 2x_i c_j x^j - x^2 c_i, \quad (3)$$

where  $a_i, \omega_{ij} = -\omega_{ji}, a$  and  $c_i$  are constants, is a solution. Which of these transformations are Poincaré transformations?

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Part IV. Section A. Relativity (continued)

Problem 2. (continued)

- b) The action for a free electromagnetic field is given by  $S = \frac{1}{4} \int d^4x F_{ij} F^{ij}$ , where  $F_{ij} = \partial_i A_j - \partial_j A_i$  is the field strength. Show that the potential  $A_i$  transforms under the transformation (1) as  $\delta A_i = (\partial_i f^j) A_j + f^j \partial_j A_i$ . Show that the action  $S$  is invariant under conformal transformations.
- c) The coupling of a free electromagnetic field to gravity is described by the same action as in the previous question but with the flat metric replaced by a general metric  $g_{ij}$ . By considering the variation of the action under  $g_{ij} \rightarrow g_{ij} + \delta g_{ij}$  derive the stress-energy tensor  $T_{ij}$  of the electromagnetic field. Calculate the trace of  $T_{ij}$ . Can you explain the result for the trace using symmetry considerations?

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Part IV. Section B. General and Atomic Physics

1. When an isotropic dielectric (for example, a gas) is placed in a strong electric field, the induced polarization is a nonlinear function of the electric field:

$$\mathbf{P} = \epsilon_o[\chi_1\mathbf{E} + \chi_3E^2\mathbf{E}] \quad (\text{MKSA}).$$

[In Gaussian units one writes  $\mathbf{P} = \chi_1\mathbf{E} + \dots$ , with the unfortunate result that  $\chi_{\text{MKSA}} = 4\pi\chi_{\text{Gaussian}}$ .] When the strong electric field is that of a laser beam, the resulting intensity dependence of the index of refraction of the gas can lead to “self-focusing” of the light into a narrow filament that propagates without diffraction. In this problem you should explore the phenomena using the simple idea of total internal reflection, and give order of magnitude estimates. You may ignore factors of  $\pi$ !

- a) Estimate the linear susceptibility  $\chi_1$  of air, which is mostly nitrogen.

Facts: The index of refraction  $n$  of liquid nitrogen at 800 nm is 1.2, and the density of liquid nitrogen is 0.8 grams/cm<sup>3</sup>.

- b) Estimate the nonlinear susceptibility  $\chi_3$  of air, supposing that when the electric field becomes strong enough to ionize nitrogen molecules,  $\chi_3E^2 \sim \chi_1$ .
- c) To study self trapping of optical beams, a laser beam of wavelength  $\lambda$  uniformly illuminates a lens of aperture  $D$  and focal length  $f$ . The laser power is adjusted so that the electric field at the focus, whose spot size is  $w_0$ , is just less than that needed to ionize air. Then, the beam can become trapped in a tube of diameter  $w_0$  because of the higher index created there by the strong field. What is the

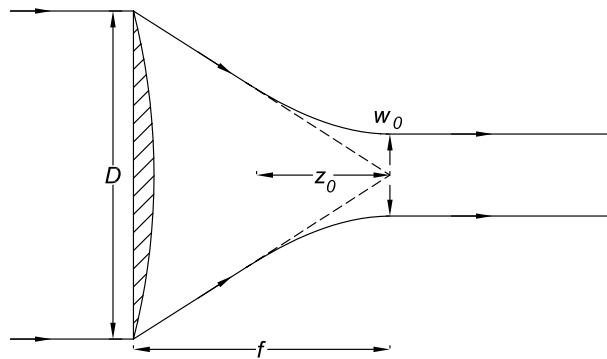
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Part IV. Section B. General and Atomic Physics (continued)

Problem 1. (continued)

minimum laser power  $P$  for such trapping to occur? If the laser pulse contained 1 J of energy, how short would the pulse need to be to achieve this power?



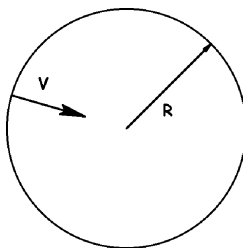
- d) If the focusing lens has an aperture  $D$  of 1 cm, what is the minimum focal length lens  $f$  for which a trapped beam will develop?

Hint: what is the relation of  $\lambda$ ,  $D$  and  $f$  to the focal spot size  $w_0$  and to the distance  $z_0$  (called the Rayleigh range) over which the beam area stays within a factor of 2 that at the focus?

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Part IV. Section B. General and Atomic Physics (continued)

2. In the phenomenon of single-bubble sonoluminescence, a water bubble of initial radius of 40 microns with zero initial radial velocity is observed to emit light when its radius collapses to about 0.5 micron radius under one atmosphere pressure. [Such bubbles can be created by strong acoustic waves that drive a cycle of expansion and compression. Here you can ignore the drive wave. The mechanism of photon emission is not well understood. It may be that the bubble interior contains some small amount of gas that is excited by the collapsing bubble and radiates photons.]
- a) During the collapse of the bubble, approximately  $10^6$  photons are emitted in the energy range 1-6 eV, with a bremsstrahlung-like spectrum of the form  $dN \sim dE/E$ , where  $E$  is the photon energy. Since water is opaque to photons above about 6 eV, presumably there are “harder” photons emitted that we cannot observe. Suppose that all the kinetic energy of the collapsing bubble is converted to photons with the spectrum  $dN \sim dE/E$ . What would be the maximum photon energy  $E_{max}$  (in eV) emitted?



- b) Since water is incompressible, as the bubble collapses the water surrounding the bubble must move radially in as well. At what radius  $R_s$  does the velocity of the inner surface  $v_s$  of the bubble reach the speed of sound in water, 1.5 km/s (2915 knots for you sailors)?

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Part IV. Section B. General and Atomic Physics (continued)

3. In its final months of operation, the world's largest circular particle accelerator, the Large Electron-Positron (LEP) collider, was pushed to its maximum energy of 100 GeV per beam, yielding collision energies of 200 GeV. Two main limitations to circular-accelerator performance are energy loss due to bremsstrahlung radiation, and the maximum value of fields that can be obtained in bending magnets. This problem explores these limitations.

Although the LEP tunnel has a circumference of 26.7 km, the effective bending radius of the dipole magnets is only 3.1 km.

- a) What is the magnetic field  $B$  in the bending magnets?
- b) What fraction of the energy of an electron is lost to synchrotron radiation during one orbit around the LEP ring at 100 GeV beam energy?

Hint: Recall the Larmor expression for the power radiated by an accelerated charge with nonrelativistic velocity.

- c) LEP will be converted to LHC, the Large Hadron Collider, and will accelerate protons instead of electrons. New magnets will be installed in the old tunnel. What is the highest possible beam energy of LHC in the future, based upon a physics estimate of the strength of materials used in the magnets?

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Part IV. Section B. General and Atomic Physics (continued)

4. In the hall outside the exam room is a double pendulum (plus some simple measuring equipment) that displays “chaotic” motion if the initial angles of displacement are sufficiently large.
- a) Carry out an experiment and describe in what sense the motion appears to be chaotic, and for what range of initial states.

For definiteness, report on initial states in which the pendula are collinear i.e., they both make the same initial angle to the vertical. Include in your report a “phase space” plot of the trajectory of the lower pendulum in the  $\theta-\dot{\theta}$  plane for a few cases of interest.

- b) The upper pendulum is made of brass ( $8.6 \text{ g/cm}^3$ ), while the lower pendulum is made of aluminum ( $2.7 \text{ g/cm}^3$ ). Estimate the moments of inertia of the upper and lower pendula. What would be the periods of small oscillation if each pendulum were separately used as a simple pendulum?
- c) The onset of chaos in many systems is associated with period doubling. What is the minimum initial angle  $\theta_0$  to the vertical (with the pendula collinear) for which the lower pendulum can undergo a motion with period approximately twice that for slightly smaller initial angles?

Hints: Sketch relevant trajectories of the lower pendulum in the  $\theta-\dot{\theta}$  plane. Since the pendula are coupled, energy can be transferred between them.

Followup: [http://www.cmp.caltech.edu/~mcc/Chaos\\_Course/Outline.html](http://www.cmp.caltech.edu/~mcc/Chaos_Course/Outline.html)