Department of Physics, Princeton University

Graduate Preliminary Examination
Part II

Tuesday, January 8, 2002
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Quantum Mechanics

1. A particle of mass m moves in the spherically symmetrical potential in 3 dimensions:

$$V(r) = \begin{cases} 0, & 0 \leq r < a, \\ -U_0, & a < r < b, \\ 0, & b < r, \end{cases}$$

where $U_0 > 0$.

What is the condition on U_0 so that there will not be any bound states?

2. In this problem you will study the way in which a wave function changes from one coordinate frame to another and then apply it to a physical problem.

   (a) A particle of mass m is described by a wave function $\psi(x,t)$ in the lab frame. What would be the wave function that describes the particle for an observer that moves with velocity v in the positive x direction?

   (**Hint:** One way to find the answer is by decomposing $\psi(x,t)$ into plane waves.)

   (b) A hydrogen atom is at rest when a neutron collides with the nucleus and causes it to move with velocity v. What is the probability that the atom will remain in its ground state after the collision? Since the proton is much heavier than the electron, you may neglect corrections of the order m_e/m_p (the ratio of the electron to proton mass). You may also assume that the collision between the proton and neutron is instantaneous.

3. Two identical neutral, spin-$\frac{1}{2}$ particles of mass M and magnetic moment μ are restricted to move on a line. The interaction between them is spin dependent and is described by the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2m} + (2\hbar^2 - S_T^2)U_0(x_1 - x_2).$$

Here $\mathbf{S}_T = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin of the system ($S_T = 0$ or 1) and U_0 is an infinite-well potential:

$$U_0(x) = \begin{cases} -\frac{\pi^2}{4ma^2}, & |x| < a, \\ \infty, & a < |x|. \end{cases}$$

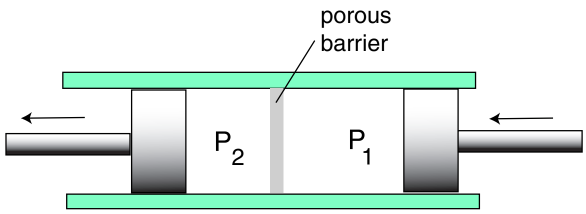
(a) Find the energy eigenstates of the system (in zero magnetic field) and their corresponding wave-functions. What is the energy E_0 of the ground state?

(b) Assume that initially the system is in the ground state and that $\hbar\omega + E_0 > 0$. To first order in perturbation theory, what is the half lifetime of the bound state in the presence of an electromagnetic plane wave with magnetic field
$B_z = B_0 \cos k(x - ct)$.
You may use the nonrelativistic approximation $ka \ll 1$ and expand to lowest order in ka.

Section B. Thermodynamics and Statistical Mechanics

1. A thermally isolated vessel containing a non-ideal gas is separated in two parts by a porous barrier. Initially all of the gas is on one side of the barrier and occupies a volume V. The gas is transferred slowly through the barrier by moving two pistons inward and outward, while keeping the pressures P_1 and P_2 fixed on both sides of the barrier. This is called a Joule-Thomson process. For an ideal gas the temperatures T_1 and T_2 before and after the process are the same. For a non-ideal gas there will be a small difference $\Delta T = T_2 - T_1$. The problem is to determine ΔT for a non-ideal gas described by the van der Waals equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT.$$



In this problem we assume that the pressure difference is small, so that after the process the volume has increased only by a small amount $\Delta V = V_2 - V_1$.

(a) Calculate the free energy $F(V, T)$ for a van der Waals gas with total specific heat C_V.

(b) Show that the enthalpy $H \equiv U + PV$ is constant for a Joule-Thomson process.
Hint: Argue that

$$\Delta T = \left(\frac{\partial T}{\partial V}\right)_H \Delta V.$$

(c) Find the enthalpy H for a van der Waals gas as a function of V and T.

(d) Show that ΔT is positive for high temperature and negative at low temperatures. The temperature $T_{\rm inv}$ at which ΔT changes sign is called the inversion temperature. Derive that

$$T_{\rm inv} = \frac{2a}{bR}\left(1 - \frac{b}{V}\right)^2.$$

2. Consider a degenerate gas of N nonrelativistic neutrons of magnetic moment μ_B in a volume V. The gas is placed in a constant magnetic field H. The problem is to determine the magnetic moment M of the neutron gas, and its susceptibility $\chi = \partial M/\partial H$ at temperature $T = 0$.

   (a) Derive integral expressions for the average number of neutrons N^+ (N^-) with spin up (down) as a function of the chemical potential μ for $T \neq 0$.

   (b) Evaluate the integrals in the limit $T \to 0$, where $\mu \to \epsilon_F$, the Fermi energy.

   (c) Express the magnetization M in terms of the Fermi energy ϵ_F. Find the condition that determines ϵ_F in terms of N and $\mu_B H$.

   (d) Use your result from c) to calculate the susceptibility χ for $\mu_B H \ll \epsilon_F$ at $T = 0$.

3. Consider an ensemble of $N \gg 1$ independent identical oscillators of natural frequency ω. Suppose there is a total of M quanta (bosons) to distribute among the ensemble. The number of *distinct* ways to do so may be shown to be

$$W(M) = \frac{(M+N-1)!}{M!(N-1)!}\,.$$

   (a) Write down the internal energy E and the entropy S of the ensemble in terms of M, N and ω.

   (b) Now suppose the system comes to equilibrium with a heat reservoir at temperature T. By minimizing an appropriate thermodynamic function, find the average distribution $n(T) = \langle M \rangle / N$.

   (c) Derive the heat capacity C_V versus T.

   (d) Verify that at equilibrium the derivative $d\langle S \rangle / dE$ gives the inverse temperature.

   (e) Derive Eq. (1). [*Hint*: Think partitions.]