

Department of Physics, Princeton University

**Graduate General Examination
Part III**

Monday, May 10, 2004

9:00 am - 12:00 noon

This part of the General Examination poses **THREE** questions on Condensed Matter Physics and **THREE** on Elementary Particles and Nuclear Physics. Answer **THREE** questions, at least **ONE** from each section.

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Condensed Matter Physics

1. Under sufficiently high pressures even hydrogen will exhibit a solid phase. Some calculations suggest that it will form an FCC lattice with one atom per unit cell.

- (a) Taking the nearest neighbor distance to be a and the hopping matrix element between neighboring sites to be $-t$, calculate the dispersion relation for the electrons (the band structure) in the tight binding approximation. Is the system metallic or insulating in this treatment?

Now consider the effects of adding back in the interactions between the electrons.

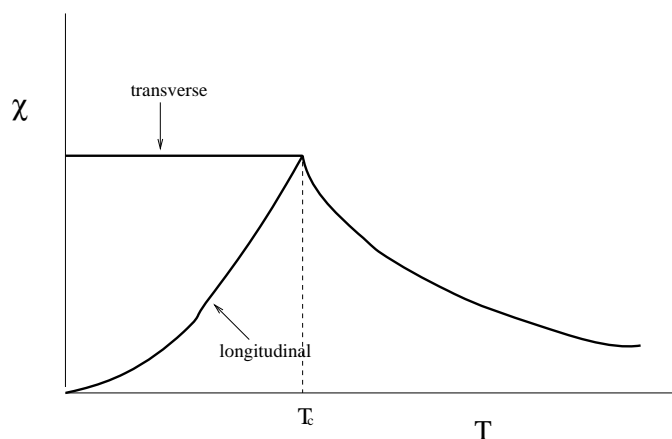
- (b) As $a \rightarrow \infty$ the system is insulating. Why? Roughly how close would we need to push the atoms in order for it to become metallic? Justify your answer but factors of order one can be ignored with impunity.

(Hint: The Thomas-Fermi screening length for an electron gas of density n is given by

$$\frac{1}{\lambda_{TF}^2} = 4 \left(\frac{3}{\pi} \right)^{1/3} \frac{n^{1/3}}{a_0},$$

where a_0 is the Bohr radius.)

2. The following “data” describe the (uniform) magnetic susceptibility of an antiferromagnet on the simple cubic lattice.



Take the Hamiltonian for this system to be

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \mathbf{B} \cdot \sum_i \mathbf{S}_i \quad (1)$$

where $\langle ij \rangle$ indicates a sum over nearest neighbor pairs of sites on the lattice, and treat the spins as classical vectors of length 1.

- Show that when $J = 0$ each spin acquires an expectation value $\mathbf{B} \Phi(B/k_B T)/B$ at temperature T , where $\Phi(x) = \coth(x) - 1/x$.
- Write the mean field equations governing the local magnetizations when $J > 0$.
- From the above equations calculate the transition temperature T_c in zero magnetic field.
- Next calculate the susceptibility for $T > T_c$.
- Explain the contrasting behaviors of the longitudinal and transverse susceptibilities as $T \rightarrow 0$.

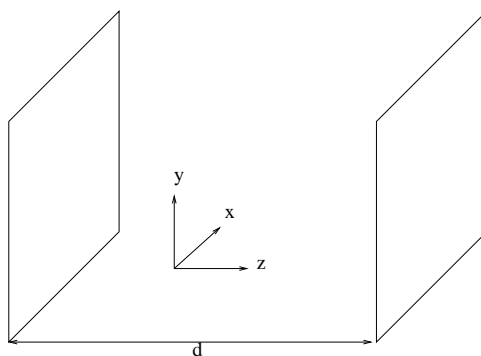
(The terms longitudinal and transverse refer to fields oriented along and perpendicular to, the axis of the local magnetizations in the low temperature phase. Also, you may find the asymptotic forms, $\Phi(x) \sim x/3$, $x \ll 1$ and $\Phi(x) \sim 1 - 1/x$, $x \gg 1$ useful.)

3. The elastic free energy for a nematic liquid crystal in the presence of an electric field \vec{E} is

$$F = \frac{1}{2} \int d^3r [K_1(\nabla \cdot \vec{n})^2 + K_2(\vec{n} \cdot (\nabla \times \vec{n}))^2 + K_3(\vec{n} \times (\nabla \times \vec{n}))^2 - \frac{\epsilon}{8\pi}(\vec{E} \cdot \vec{n})^2]$$

where \vec{n} is a unit vector, $\vec{n} \cdot \vec{n} = 1$, describing the direction of the long axis of the molecules.

Consider a nematic sandwiched between two parallel plates, a distance d apart, with the boundary condition that $\vec{n} = \hat{y}$ at the plates (see Figure).



- (a) What is $\vec{n}(z)$ in the region between the plates at zero electric field?
- (b) Now consider the situation in which an electric field $\vec{E} = E\hat{x}$ is applied along the x direction. Show that there is a critical electric field, E_c , below which the molecules remain in the state you found in the previous part and above which they start to rotate towards the field (i.e. towards \hat{x}). What is the value of this field?

(Hint: As a first step show that, for small tilts of \vec{n} , of the elastic constants K_i only K_2 contributes to F .)

Section B. Elementary Particles and Nuclear Physics

1. Spectroscopy of charmed baryons

The figure shows the spectrum of the lightest baryons with charm.

Mass (MeV/c ²)				J ^P
2520	$\underline{\Sigma_c^{*0}}$	$\underline{\Sigma_c^{*+}}$	$\underline{\Sigma_c^{*++}}$	$\frac{3}{2}^+$
2455	$\underline{\Sigma_c^0}$	$\underline{\Sigma_c^+}$	$\underline{\Sigma_c^{++}}$	$\frac{1}{2}^+$
2285	$\underline{\Lambda_c}$			$\frac{1}{2}^+$

In the constituent quark model, these states are interpreted as non-relativistic bound states with quark content (cqq), where q is a light quark (u or d).

- (a) Assuming that the quarks are in their orbital ground state, show that this model accounts precisely for the existence of the states in the figure. Write down all the allowed wavefunctions, indicating all quantum numbers. State your assumptions.
- (b) The main contribution to mass splittings comes from the spin-spin (“hyperfine”) interactions between constituent quarks. This is expressed by the mass formula

$$M = \sum_{i=1}^3 m_i + A \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{m_i m_j},$$

where m_i is the mass and \vec{S}_i the spin operator for the i -th constituent quark. A is a phenomenological parameter. Compute the mass splittings $M_{\Sigma_c} - M_{\Lambda_c}$ and $M_{\Sigma_c^*} - M_{\Sigma_c}$ in terms of $m_u = m_d$, m_c and A . Use the data in the figure to estimate m_u/m_c .

- (c) Based on your knowledge of selection rules, what are the dominant decays of Σ_c and Σ_c^* ?

2. Rutherford scattering

For a scattering process $AB \rightarrow CD$ the differential cross section is given by

$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ$$

where

$$dQ = (2\pi)^4 \delta^{(4)}(p_C + p_D - p_A - p_B) \frac{d^3 \mathbf{p}_C}{(2\pi)^3 2E_C} \frac{d^3 \mathbf{p}_D}{(2\pi)^3 2E_D}$$

is the phase space factor, F is the incident flux, and \mathcal{M} is the scattering amplitude. In this problem we consider the tree-level scattering amplitude due to photon exchange (neglecting the spin of the incident particles),

$$\mathcal{M} = -Z_A e (p_A + p_C)^\mu \left(\frac{g_{\mu\nu}}{(p_A - p_C)^2} \right) Z_B e (p_B + p_D)^\nu.$$

The variables p , \mathbf{p} , E , and Ze refer to the particle 4- and 3-momentum, energy, and charge, respectively.

- Calculate the differential cross section $\frac{d\sigma}{d\Omega}$ for alpha particles scattering on a much heavier nucleus of charge Ze . Take the non-relativistic limit of your answer.
- A parallel beam of alpha particles with non-relativistic kinetic energy T is incident on a gold foil of thickness t . A small square detector of area a is placed at a distance d from the foil at an angle of 45 degrees from the beam axis. If the incident beam rate is I (alpha particles per second) what is the rate at which scattered alphas hit the detector? State your answer in terms of the quantities given and any other needed parameters.

3. Physics of D mesons

In this problem we look at some aspects of the physics of D mesons. The information given at the end may be useful.

- (a) Recall that the widths for π^+ and μ^+ decays are

$$\Gamma_\pi = \frac{G_F^2 f_\pi^2 m_\pi m_\mu^2}{8\pi} \left(1 - \frac{m_\mu^2}{m_\pi^2}\right), \quad \Gamma_\mu = \frac{G_F^2 m_\mu^5}{190\pi^3},$$

where $f_\pi = 131$ MeV. Give an order-of-magnitude estimate for the branching ratio for $D^+ \rightarrow \mu^+ \nu_\mu$. Assume that any unknown constants for the D^+ are the same as for the π^+ .

- (b) Write the weak decay *amplitude* (not the rate) for the decay $D^+ \rightarrow \bar{K}^0 \pi^+$ in terms of the amplitudes for the decays $D^0 \rightarrow K^- \pi^+$ and $D^0 \rightarrow \bar{K}^0 \pi^0$.
- (c) There are two mechanisms for the transition $D^0 \rightarrow K^+ \pi^-$, one direct and the other indirect (through mixing with \bar{D}^0). Draw the relevant Feynman diagrams for the two mechanisms. Estimate the rate for the direct decay relative to the rate for $D^0 \rightarrow K^- \pi^+$.

Some potentially useful information:

$$\begin{array}{lll} \bar{K}^0 = \bar{d}s & K^+ = u\bar{s} & D^+ = c\bar{d} \quad \bar{D}^0 = \bar{c}u \\ K^- = \bar{u}s & K^0 = d\bar{s} & D^0 = c\bar{u} \quad D^- = \bar{c}d \end{array}$$

$$m_\mu = 106 \text{ MeV} \quad m_{\pi^\pm} = 140 \text{ MeV} \quad m_{\pi^0} = 135 \text{ MeV}$$

$$\begin{array}{ll} m_{K^\pm} = 494 \text{ MeV} & m_{K^0} = 498 \text{ MeV} \\ m_{D^\pm} = 1869 \text{ MeV} & m_{D^0} = 1865 \text{ MeV} \end{array}$$

(See also Table on next page.)