Department of Physics, Princeton University

Graduate Preliminary Examination
Part II

Friday, May 7, 2002
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Quantum Mechanics

1. **Hydrogen molecule**

   Consider the neutral hydrogen molecule H_2. Write down the hamiltonian keeping only the kinetic energy terms and the Coulomb interactions of all the constituents and omitting terms which cause fine and hyperfine structure.

   (a) What is its degeneracy of the ground state? Give all quantum numbers and symmetries of the ground state(s), including the electron and the proton spin degrees of freedom.

   (b) What is the degeneracy, and what are all the quantum numbers of the first excited state of this H_2 molecule? Explain.

   (c) What is the energy difference between ground and first excited states? Estimate it first through a formula, in terms of properties of the molecule's ground state, and then in electron-Volts (eV).

2. **Positronium in a magnetic field**

   Consider the spin degrees of freedom of a two-particle system, one with spin S and the other with spin 1/2. The Hamiltonian is

   $$H = a\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \ , \tag{1}$$

   with a a constant. Here $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ stand for the vector spin operators of particle 1 and 2, respectively.

   (a) Calculate the eigenvalues of H. What are their multiplicities?

   (b) Consider now the special case, corresponding to the spin degrees of freedom of positronium, where both spins are of $S = 1/2$. What are the eigenvalues and corresponding eigenstates of H?

   (c) When the positronium is placed in a magnetic field, oriented in the z-direction, the Hamiltonian becomes

   $$H_B = a\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + b(S_1^z - S_2^z) \ , \tag{2}$$

   where b is a constant. Describe the multiplicities of the resulting spectrum, and calculate the eigenvalues and eigenvectors of H_B.

3. A beam of particles of mass m and energy E propagates along the z axis of a coordinate system, and scatters from the cubic potential

$$V = \begin{cases} v & \text{if } |x| \leq \text{L}, \ |y| \leq \text{L}, \text{ and } |z| \leq \text{L}, \\ 0 & \text{otherwise} \end{cases}$$

where v is a small constant energy.

(a) Use the Born approximation to find an explicit formula for the scattering cross section $\sigma = \sigma(\theta, \phi)$ as a function of the angles θ and ϕ.
Recall that spherical coordinates of a point in space (r, θ, ϕ) are related to Cartesian coordinates (x, y, z) by $x = r sin\theta\, cos\phi$, $y = r sin\theta\, sin\phi$ and $z = r cos\theta$. The Born approximation is easy to evaluate in one coordinate system and hard in the other.

(b) Under what circumstances is this approximation for the scattering cross section valid? Explain.

Section B. Statistical Mechanics and Thermodynamics

1. Experimentally, the equation of state (internal energy, U, and pressure, p) of a substance was found to be:

$$U = Vu(T)\,, \qquad p = \alpha\, u(T)\,,$$

where V is the volume, T is the temperature, α is a constant and $u(T)$ is an unknown function.

(a) Determine the relation of T and V in an adiabatic process. If the process is instead isothermal, how much heat is exchanged with the heat bath?

(b) Sketch the $p - V$ diagram for a Carnot cycle (2 adiabats, 2 isotherms) using this substance. Express its efficiency in terms of the function $u(T)$.

(c) Explain why the above efficiency should equal $1 - T_c/T_h$, where T_c and T_h are the temperatures of the low and high temperature isotherms in the Carnot cycle.

(d) What conclusions can you draw about the function $u(T)$, corresponding to the given value of α?

(e) Give a physical system that shows this behavior.

2. Consider waves on a liquid surface where the restoring force is produced by the surface tension. Assume there is a single polarization and the dispersion relation is

$$\omega^2 = \frac{\gamma}{\rho} k^3 \,,$$

where γ is the surface tension of the liquid, ρ is its density, ω is the frequency of the waves and k is their wavenumber. Our goal is to find the contribution of these waves to the heat capacity of the liquid.

(a) If the surface is in equilibrium at temperature T, what is the average energy of a wave with frequency ω? (Ignore the $\hbar\omega/2$ zero point energy.)

(b) At low temperatures what are the energy per unit area and heat capacity per unit area of these surface waves? You may leave dimensionless integrals in your answer.

(c) What can you say about the high temperature heat capacity per unit area?

3. Consider a binary mixture of atoms labeled A and B. There are $N \gg 1$ atoms of which xN are of type A and $(1-x)N$ are of type B. The atoms occupy equidistant sites along a line. Ignoring the kinetic energy, the statistical mechanics is governed by the potential energy which is determined as follows: two neighboring A atoms or two neighboring B atoms contribute $-\epsilon$ while a pair of neighboring ABs contribute $-\epsilon/2$. A sample arrangement may look like

$$\cdots \mathrm{AAABABBBAABAAA} \cdots$$

   (a) What is the average size of a cluster of A atoms at temperature $T = \infty$ in the limit $N \to \infty$?

   (b) What is the average size of a cluster of A atoms at $T = 0$ in the limit $N \to \infty$?

   (c) Calculate an estimate for the free energy of the mixture regarding the atoms to be independently and randomly distributed.

   (d) Does the above value provide a variational bound on the free energy? (Upper, or lower bound?)

   (e) Within this approximation, estimate the phase transition temperature at $x = 1/2$.

   (f) Actually, this system has no phase transition at $T > 0$. Explain why.